THE CHINESE UNIVERSITY OF HONG KONG
Department of Mathematics

# MATH4060 Complex Analysis
# Homework 1 Suggested Solutions

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1. (Exercise 4.2 of [SS03]) If $f \in \mathfrak{F}_a$ with $a > 0$, then for any positive integer $n$ one has $f^{(n)} \in \mathfrak{F}_b$ whenever $0 \leq b < a$.

**Solution.** Let $a > 0$ and denote by $S_a$ the horizontal strip

$$S_a := \{z \in \mathbb{C} : |\mathrm{Im}(z)| < a\}.$$

Then recall that the condition that $f \in \mathfrak{F}_a$ means that

(a) $f$ is holomorphic in $S_a$,

(b) there exists a constant $A > 0$ such that

$$|f(x+iy)| \leq \frac{A}{1+x^2} \quad \text{for all } x \in \mathbb{R} \text{ and } |y| < a.$$

Let $0 \leq b < a$ and consider $f^{(n)}$. Since $S_b$ is open for all $0 \leq b < a$, the equivalence of the holomorphic property and analyticity in an open set (i.e. Theorem 2.6, Corollary 2.7, and Theorem 4.4 of [SS03]) gives that $f^{(n)}$ is also holomorphic in $S_b$ for each $n \in \mathbb{N}$. In fact, we actually have that $f^{(n)}$ is holomorphic in $S_a$ for each $n \in \mathbb{N}$.

It remains to check that there is a constant $A_n > 0$ such that

$$\left|f^{(n)}(x+iy)\right| \leq \frac{A_n}{1+x^2} \quad \text{for all } x \in \mathbb{R} \text{ and } |y| < b.$$

Let $\delta := a - b > 0$ and consider the disk of radius $\delta$ centred at $w \in S_b$ (thus $D_\delta(w) \subset S_a$ still. Then by the Cauchy integral formula, we have

$$\begin{aligned}
\left|f^{(n)}(w)\right| &= \frac{n!}{2\pi}\left|\int_{\partial D_\delta(w)} \frac{f(z)}{(z-w)^{n+1}}dz\right| \\
&= \frac{n!}{2\pi}\left|\int_0^{2\pi} \frac{f(w+\delta e^{i\theta}\delta e^{i\theta}}{(\delta e^{i\theta})^{n+1}}d\theta\right| \\
&\leq \frac{n!}{2\pi}\int_0^{2\pi} \frac{\left|f(w+\delta e^{i\theta})\right|}{\delta^n}d\theta \\
&\leq \frac{n!}{2\pi\delta^n}\int_0^{2\pi} \frac{A}{1+\left(\mathrm{Re}(w+\delta e^{i\theta})\right)^2}d\theta.
\end{aligned}$$

One can check that when $|\mathrm{Re}(w)| > 2\delta$, $\left(\mathrm{Re}(w)+\delta e^{i\theta}\right)^2 \geq \dfrac{\mathrm{Re}(w)^2}{2} - \delta^2$, and so

$$\frac{A}{1+\left(\mathrm{Re}(w+\delta e^{i\theta})\right)^2} \leq \frac{A}{1+\frac{\mathrm{Re}(w)^2}{2}-\delta^2} \leq \frac{A}{1+\frac{\mathrm{Re}(w)^2}{4}} \leq \frac{4A}{1+\mathrm{Re}(w)^2},$$

and when $|\mathrm{Re}(w)| \le 2\delta$, we instead have

$$\frac{A}{1+(\mathrm{Re}(w+\delta e^{i\theta}))^2} \le \frac{A(1+4\delta^2)}{1+\mathrm{Re}(w)^2}.$$

So taking $A_n := \max\left\{\frac{4An!}{2\pi\delta^n}, \frac{A(1+4\delta^2)n!}{2\pi\delta^n}\right\}$, we are done. ◀

2. (Exercise 4.4 of [SS03]) Suppose $Q$ is a polynomial of degree $\ge 2$ with distinct roots, none lying on the real axis. Calculate

$$\int_{-\infty}^{\infty} \frac{e^{-2\pi i x\zeta}}{Q(x)}dx, \quad \zeta \in \mathbb{R}$$

in terms of the roots of $Q$. What happens when several roots coincide? [Hint: Consider separately the cases $\zeta < 0, \zeta = 0$, and $\zeta > 0$. Use residues.]

**Solution.** Write $Q(z) = a_n z^n + a_{n-1}z^{n-1} + \cdots + a_1 z + a_0$ with $a_n \neq 0$. For $|z| = R > 0$ large enough depending on $|a_n|$, we have the estimate

$$\begin{aligned}
|Q(z)| &= |a_n z^n + a_{n-1}z^{n-1} + \cdots + a_1 z + a_0| \\
&\ge |a_n||z|^n - |a_{n-1}||z|^{n-1} - \cdots - |a_1||z| - |a_0| \\
&= R^n\left(|a_n| - \frac{|a_{n-1}|}{R} - \cdots - \frac{|a_1|}{R^{n-1}} - \frac{|a_0|}{R^n}\right) \\
&\ge \frac{|a_n|}{2}R^n \\
&\ge \frac{|a_n|}{2}R^2
\end{aligned}$$

since $n \ge 2$.

Suppose $Q(z)$ has distinct roots $\alpha_1, \dots, \alpha_\ell$ with multiplicities $m_1, \dots, m_\ell$ such that $\sum_{k=1}^{\ell} m_k = n$, that is, we can write $Q(z) = (z-\alpha_1)^{m_1}\cdots(z-\alpha_k)^{m_k}\cdots(z-\alpha_\ell)^{m_\ell}$. Without loss of generality, we can order the roots so that the roots numbered $1, \dots, \ell_0$ lie in the upper half-plane and the roots numbered $\ell_0 + 1, \dots, \ell$ lie in the lower half-plane for some $1 \le \ell_0 \le \ell$. Note that there is an $R > 0$ large enough so that all the roots have modulus less than $R$, i.e., they all lie within the disk of radius $R$.

We first handle the case where $\zeta \le 0$. Let $\gamma_R := I_R \cup C_R^+$ be the contour composed of the interval $I_R = [-R, R]$ and the upper half-circle of radius $R$ in the counterclockwise orientation. Then the residue theorem gives

$$\int_{C_R^+} \frac{e^{-2\pi i z\zeta}}{Q(z)}dz + \int_{-R}^{R} \frac{e^{-2\pi i x\zeta}}{Q(x)}dx = \int_{\gamma_R} \frac{e^{-2\pi i z\zeta}}{Q(z)}dz = 2\pi i \sum_{k=1}^{\ell_0} \mathrm{res}_{z=\alpha_k} \frac{e^{-2\pi i z\zeta}}{Q(z)}.$$

Denote by $Q_k(z) = \dfrac{e^{-2\pi i z\zeta}}{(z-\alpha_1)^{m_1}\cdots(z-\alpha_{k-1})^{m_{k-1}}(z-\alpha_{k+1})^{m_{k+1}}\cdots(z-\alpha_\ell)^{m_\ell}}$, i.e. $\frac{e^{-2\pi i z\zeta}}{Q(z)} = \frac{Q_k(z)}{(z-\alpha_k)^{m_k}}$. Then we have

$$\mathrm{res}_{z=\alpha_k} \frac{e^{-2\pi i z\zeta}}{Q(z)} = \frac{Q_k^{(m_k-1)}(\alpha_k)}{(m_k-1)!}$$

where $Q_k^{(m_k-1)}(\alpha_k)$ denotes the $(m_k-1)$-th derivative of $Q_k(z)$ (see Section 80 "Residues at Poles" of Brown and Churchill's "Complex Variables and Applications", ninth edition).

Using the estimate above, we have that

$$\begin{aligned}\left|\int_{C_R^+} \frac{e^{-2\pi iz\zeta}}{Q(z)}dz\right| &\leq \int_0^\pi \frac{e^{2\pi R\zeta\sin\theta}}{\frac{|a_n|}{2}R^2}Rd\theta \\ &\leq \int_0^\pi \frac{1}{\frac{|a_n|}{2}R}d\theta \\ &= \frac{\pi}{\frac{|a_n|}{2}R}\end{aligned}$$

which vanishes as we take $R \to +\infty$. So taking $R \to +\infty$, we have that

$$\int_{-\infty}^{\infty} \frac{e^{-2\pi ix\zeta}}{Q(x)}dx = 2\pi i \sum_{k=1}^{\ell_0} \frac{Q_k^{(m_k-1)}(\alpha_k)}{(m_k-1)!}.$$

For the case where $\zeta > 0$, a similar argument using the contour $\tilde{\gamma}_R = \tilde{I}_R \cup C_R^-$ where $\tilde{I}_R$ is the straight line segment going from $-R$ to $R$ and $C_R^-$ is the lower half-circle of radius $R$ in the clockwise orientation yields

$$\int_{-\infty}^{\infty} \frac{e^{-2\pi ix\zeta}}{Q(x)}dx = -2\pi i \sum_{k=\ell_0+1}^{\ell} \frac{Q_k^{(m_k-1)}(\alpha_k)}{(m_k-1)!}.$$

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3. (Exercise 4.6 of [SS03]) Prove that

$$\frac{1}{\pi}\sum_{n=-\infty}^{\infty} \frac{a}{a^2+n^2} = \sum_{n=-\infty}^{\infty} e^{-2\pi a|n|}$$

whenever $a > 0$. Hence show that the sum equals $\coth \pi a$.

**Solution.** Let $f(n) = \frac{1}{\pi}\frac{a}{a^2+n^2}$. Then by the Poisson summation formula, we have

$$\sum_{n\in\mathbb{Z}} f(n) = \sum_{n\in\mathbb{Z}} \hat{f}(n)$$

where $\hat{f}(\zeta) = \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{a}{a^2+x^2}e^{-2\pi ix\zeta}d\zeta$. Then, using the result of Exercise 4.3 (also covered in tutorial), we know that

$$\hat{f}(\zeta) = \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{a}{a^2+x^2}e^{-2\pi ix\zeta}d\zeta = e^{-2\pi a|\zeta|}$$

and hence

$$\frac{1}{\pi}\sum_{n=-\infty}^{\infty}\frac{a}{a^2+n^2} = \sum_{n=-\infty}^{\infty} e^{-2\pi a|n|} = 1+2\sum_{n=1}^{\infty} e^{-2\pi an}$$
$$= 1+2\left(\frac{e^{-2\pi a}}{1-e^{-2\pi a}}\right) = \frac{1+e^{-2\pi a}}{1-e^{-2\pi a}} = \coth \pi a$$

as required. ◀

4. (Exercise 4.9 of [SS03]) Here are further results similar to the Phragmén-Lindelöf theorem.

   (a) Let $F$ be a holomorphic function in the right half-plane that extends continuously to the boundary, that is, the imaginary axis. Suppose that $|F(iy)| \leq 1$ for all $y \in \mathbb{R}$, and
   $$|F(z)| \leq Ce^{c|z|^\gamma}$$
   for some $c, C > 0$ and $\gamma < 1$. Prove that $|F(z)| \leq 1$ for all $z$ in the right half-plane.

   (b) More generally, let $S$ be a sector whose vertex is the origin, and forming an angle of $\pi/\beta$. Let $F$ be a holomorphic function in $S$ that is continuous on the closure of $S$, so that $|F(z)| \leq 1$ on the boundary of $S$ and
   $$|F(z)| \leq Ce^{c|z|^\alpha} \text{ for all } z \in S$$
   for some $c, C > 0$ and $0 < \alpha < \beta$. Prove that $|F(z)| \leq 1$ for all $z \in S$.

**Solution.** (a) One can see that (a) follows from (b), so we only show (b).

(b) Fix $\gamma$ with $\alpha < \gamma < \beta$. After a rotation, without loss of generality we can assume $S$ is the set
$$S = \left\{ z \in \mathbb{C} : -\frac{\pi}{2\beta} < \arg(z) < \frac{\pi}{2\beta} \right\}.$$
Let $\varepsilon > 0$ be given. Define
$$G_\varepsilon(z) = F(z)e^{-\varepsilon z^\gamma} = F(z)e^{-\varepsilon \exp(\gamma \log(z))}$$
and note that we can choose a branch of the logarithm on $S$ that is well-defined and holomorphic.

Note that on the boundary of $S$, we have that
$$|G_\varepsilon(z)| = |F(z)e^{-\varepsilon z^\gamma}| \leq e^{-\varepsilon|z|^\gamma} \leq 1$$
since $|F(z)| \leq 1$ on the boundary of $S$.

Let $z \in S$ and write $z = Re^{i\theta}$ for some $R > 0$ and $-\dfrac{\pi}{2\beta} \leq \theta\dfrac{\pi}{2\beta}$. Then we write $z^\gamma = R^\gamma \cos(\gamma\theta) + iR^\gamma \sin(\gamma\theta)$ and we have

$$\begin{aligned}
|G_\varepsilon(z)| &= |F(z)e^{-\varepsilon z^\gamma}| \leq |F(z)||e^{-\varepsilon z^\gamma}| \leq |F(z)|e^{-\varepsilon R^\gamma \cos(\gamma\theta)}|e^{-i\varepsilon R^\gamma \sin(\gamma\theta)}| \\
&\leq |F(z)|e^{-\varepsilon R^\gamma \cos(\gamma\theta)} \leq |F(z)|e^{-\varepsilon R^\gamma \cos\left(\frac{\pi\gamma}{\beta}\right)} \\
&= |F(z)|e^{-cR^\alpha} e^{R^\gamma\left(c - \varepsilon R^{\gamma-\alpha} \cos\left(\frac{\pi\gamma}{\beta}\right)\right)} \\
&\leq Ce^{R^\gamma\left(c - \varepsilon R^{\gamma-\alpha} \cos\left(\frac{\pi\gamma}{\beta}\right)\right)} \to 0 \text{ as } R \to +\infty
\end{aligned}$$

where we used the assumption on $F$ and the fact that $\gamma > \alpha$. Since $G_\varepsilon$ vanishes at infinity, the maximal $M = \max_{\overline{S}} |G_\varepsilon(z)|$ must be achieved on some point $a$. If $a \in S$, the maximum modulus principle implies that $G_\varepsilon$ is constant and therefore $|G_\varepsilon(z)| \leq 1$ in $\overline{S}$ since $G_\varepsilon$ is continuous to the boundary and we have that $|G_\varepsilon(z)| \leq 1$ on the boundary as shown above. If however $a \in \partial S$, we already know that $|G_\varepsilon(z)| \leq 1$ there and hence we know that $|G_\varepsilon(z)| \leq 1$ in $\overline{S}$. In either case, we have that $|G_\varepsilon(z)| \leq 1$ on $\overline{S}$. Taking $\varepsilon \to 0^+$ and arguing by continuity, we obtain the desired result.

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# References

[SS03] Elias M. Stein and Rami Shakarchi. *Complex Analysis*. Princeton University Press, 2003.